Symmetric Functions and the Affine Symmetric Group — Anakin Dey
Last Edited on 5/26/26 at 16:27

1 Symmetric Functions

First we review the construction of the ring of symmetric functions. We will assume basic notations for partitions and (skew) tableaux following [Ful96].

Let $\mathbb{Z}[x_1, x_2, \ldots]$ be the polynomial ring over $\mathbb{Z}$ in countably many variables $\{x_i\}_{i\geq 1}$.

Definition 1: Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be any partition with length $\ell(\lambda) \leq n$. Then the *monomial symmetric function* m_λ is defined by

$$m_\lambda = \sum_\alpha \overline{x}^\alpha,$$

where the summation is taken over all permutations α of λ.

Definition 2: Let $\Lambda_k \subseteq \mathbb{Z}[x_1, x_2, \ldots]$ be the ring generated by all of the monomial symmetric functions m_λ as $\lambda \vdash k$ ranges over all partitions of k. We have that Λ^k is the free $\mathbb{Z}$-module of rank $p(k)$, where $p(k)$ is the number of partitions of k. The *ring of symmetric functions* Λ is the direct sum $\bigoplus_{k\geq 0} \Lambda^k$. This is a graded ring where each x_i has degree one. To obtain the ring of symmetric functions Λ_A over rings A other than $\mathbb{Z}$, repeating the above construction gives $\Lambda_A \simeq \Lambda \otimes_{\mathbb{Z}} A$.

As an aside, we also give the construction of Λ as a limit of graded rings following [Mac95]. For $n \geq 0$, let $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ be the ring of symmetric polynomials over $\mathbb{Z}$ in n variables. For $k \geq 0$, let Λ_n^k be the k-th graded component of Λ_n, where each x_i has degree one. For $m \geq n$, we have a restriction morphism $\rho_{m,n} \colon \Lambda_m \to \Lambda_n$ sending $x_{n+1}, \ldots, x_m$ to zero and $x_1, \ldots, x_n$ to themselves. Furthermore, $\rho_{m,n}$ respects the grading and so we have $\rho_{m,n}^k \colon \Lambda_m^k \to \Lambda_n^k$ for all $k \geq 0$ and all $m \geq n \geq 0$.

The *ring of symmetric functions* Λ has as the k-th graded component the inverse limit $\Lambda^k := \varprojlim_n \Lambda_n^k$ and we define $\Lambda := \bigoplus_{k\geq 0} \Lambda^k$. Note that this limit is over *graded rings* and so we do not have elements such as $\prod_{i=1}^\infty (1 + x_i)$. To obtain the ring of symmetric functions Λ_A over rings A other than $\mathbb{Z}$, repeating the above construction gives $\Lambda_A \simeq \Lambda \otimes_{\mathbb{Z}} A$.

1.1 Alternative Bases for the Ring of Symmetric Functions

Definition 3:

- For $r \geq 0$, the r-th *elementary symmetric function* e_r is defined by

$$e_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} = m_{(1^r)}.$$

  For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$. The *Fundamental Theorem of Symmetric Functions* states that over finitely many variables, say $x_1, \ldots, x_n$, the elementary symmetric functions $e_1, \ldots, e_n$ are algebraically independant and $\Lambda_n \simeq \mathbb{Z}[e_1, \ldots, e_n]$. For a clean proof of this fact due to van der Waerden, see [Stu08, Theorem 1.1.1].

- For $r \geq 0$, the r-th *complete homogeneous symmetric function* h_r is defined by

$$h_r = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r} = \sum_{\lambda \vdash r} m_\lambda.$$

  For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$.

- For $r \geq 0$, the r-th *power sum symmetric function* p_r is defined by

$$p_r = \sum_{i \geq 1} x_i^r = m_{(r)}.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$.

1.2 Schur Functions We follow [Sta23, Chapter 7] for the exposition on Schur functions.

Definition 4: For a Young tableaux T, its *type* is the vector $\alpha = (\alpha_1, \alpha_2, \ldots)$ where α_i is the number of entries of T labeled with i. If T has type α, write $\overline{x}^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For a partition λ, the *Schur function* $s_\lambda(\overline{x})$ is defined by

$$s_\lambda(\overline{x}) = \sum_{\text{SSYT } T \text{ of shape } \lambda} \overline{x}^T$$

where the summation is taken over all semistandard Young tableau of shape λ.

Note that the Schur functions also form a basis of Λ.

Proposition 5: Let

$$a_\delta = \det\left(x_i^{n-j}\right)_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

be the *Vandermonde determinant*. For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the *alternant* $a_{\lambda+\delta}$ is the determinant

$$a_{\lambda+\delta} = \det\left(x_i^{\lambda_j+n-j}\right)_{1 \leq i,j \leq n}.$$

Then $s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$.

Proposition 6: For a partition λ, we have that

$$s_\lambda(\overline{x}) = \det\left(h_{\lambda_i+j-i}(\overline{x})\right)_{1 \leq i,j \leq n},$$
$$s_{\lambda'}(\overline{x}) = \det\left(e_{\lambda_i+j-i}(\overline{x})\right)_{1 \leq i,j \leq n},$$

the *Jacobi-Trudi* and *Nägelsbach-Kostka* identities respectively.

In particular we have that $e_r = s_{(1^r)}$ and $h_r = s_{(r)}$.

Theorem 7 (Pieri Rules): For a partition λ and an integer $r \geq 0$, we have that

$$h_r(\overline{x}) \cdot s_\lambda(\overline{x}) = \sum_\mu s_\mu$$

where the summation over μ is over all partitions μ formed by added r boxes to the diagram of λ such that no two boxes are added to the same column.

We also have that

$$e_r(\overline{x}) \cdot s_\lambda(\overline{x}) = \sum_\mu s_\mu$$

where the summation over μ is over all partitions μ formed by added r boxes to the diagram of λ such that no two boxes are added to the same row.

1.3 An Involution and Inner Product

Definition 8: We have a map $\omega\colon \Lambda \to \Lambda$ given by $\omega(e_\lambda) = h_\lambda$.

Proposition 9: One can check that $\omega(h_\lambda) = e_\lambda$ so that ω is an involution. Furthermore, we have that $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$ and $\omega(s_\lambda) = s_{\lambda'}$.

Definition 10: We define an inner product $\langle \cdot, \cdot \rangle$ on Λ by making the two bases $\{ m_\lambda \}_\lambda$ and $\{ h_\mu \}_\mu$ dual: $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$ for all partitions λ, μ.

Proposition 11: Let $z_\lambda = \prod_k k^{n_k(\lambda)} \cdot n_k(\lambda)!$ where $n_k(\lambda)$ is the number of parts of λ of size k. Then $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$. We also have that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ so that the Schur functions form an orthonormal basis of Λ.

Proposition 12: The involution ω is self-adjoint: $\langle f, \omega(g) \rangle = \langle \omega(f), g \rangle$. Thus, we have that $\langle \omega(f), \omega(g) \rangle = \langle f, \omega^2(g) \rangle = \langle f, g \rangle$ for all $f, g \in \Lambda$.

1.4 Skew-Schur Functions and Skewing

Definition 13: For two partitions λ, μ, the *skew Schur function* $s_{\lambda/\mu}(\overline{x})$ is defined by

$$s_{\lambda/\mu}(\overline{x}) = \sum_{\text{SSYT } T \text{ of shape } \lambda/\mu} \overline{x}^T$$

where the summation is taken over all semistandard Young tableau of shape λ/μ.

Proposition 14: For partitions λ, μ, we have that

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \det\left(h_{\lambda_i - \mu_j + j - i}(\overline{x})\right)_{1 \le i,j \le n},$$
$$s_{\lambda'/\mu'}(x_1, \ldots, x_n) = \det\left(e_{\lambda_i - \mu_j + j - i}(\overline{x})\right)_{1 \le i,j \le n}.$$

Definition 15: For any symmetric function $f \in \Lambda$, the *skewing operator* $f^\perp\colon \Lambda \to \Lambda$ is defined by the property that for all $g, h \in \Lambda$ we have that $\langle f \cdot g, h \rangle = \langle g, f^\perp(h) \rangle$.

The discussion on skewing follows [GR26, Chapter 2].

Example 16: We have that $s_\mu^\perp(s_\lambda) = s_{\lambda/\mu}$.

Example 17: We have "dual versions" of the Pieri rules. For a partition λ and an integer $r \ge 0$, we have that

$$h_r^\perp(s_\lambda(\overline{x})) = \sum_\mu s_\mu$$

where the summation over μ is over all partitions μ such that λ/μ is a *horizontal strip* meaning no two boxes in the skew diagram are in the same column. We also have that

$$e_r^\perp(s_\lambda(\overline{x})) = \sum_\mu s_\mu$$

where the summation over μ is over all partitions μ such that λ/μ is a *vertical strip* meaning no two boxes in the skew diagram are in the same row.

2 The Affine Symmetric Group

We give multiple presentations of the affine symmetric group $\widetilde{\mathfrak{S}}_n$. This mostly follows [Lam+14, Chapter 1, §2] but [BB05, Chapter 8, §3] is another phenomenal source.

Definition 18: The *affine symmetric group* $\widetilde{\mathfrak{S}}_n$ is the Coxeter group generated by $\{s_0, \dots, s_{n-1}\}$ such that

$$\begin{aligned} s_i^2 &= e \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i \text{ for } i - j \neq 0, 1, n-1 \pmod{n}, \end{aligned}$$

where all of the above indices are taken $\pmod{n}$. Thus, we have a notion of the length ℓ of a word in $\widetilde{\mathfrak{S}}_n$.

Equivalently, $\widetilde{\mathfrak{S}}_n$ is the group of permutations $w\colon \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$, $w(i+kn) = w(i) + kn$ and $\sum_{i=1}^n w(i) = \sum_{i=1}^n i$. Thus, w is completely determined by $w(1), \dots, w(n)$. Each s_i can be identified with the permutation swapping $i + kn$ and $i + kn + 1$ for all $k \in \mathbb{Z}$.

Since w is determined by $w(1), \dots, w(n)$, we express $w = [w(1), \dots, w(n)]$ in *window notation*. We have an action of the s_i on a window $[w_1, \dots, w_n]$:

$$\begin{aligned} [w_1, \dots w_n] \cdot s_i &= [w_1, \dots, w_{i+1}, w_i, \dots, w_n], && 1 \leq i \leq n-1, \\ [w_1, \dots w_n] \cdot s_0 &= [w_n - n, w_2, \dots, w_{n-1}, w_1 + n]. \end{aligned}$$

2.1 Affine Grassmannian Elements

Proposition 19: For $w \in \widetilde{\mathfrak{S}}_n$, the length $\ell(w)$ is given by

$$\#\left\{ (i,j) \in [n] \times [n] \,\middle|\, i < j, w_i > w_j \right\} + \sum_{1 \leq i < j \leq n} \left\lfloor \frac{|w_i - w_j|}{n} \right\rfloor.$$

Definition 20: We have a copy of $\mathfrak{S}_n \subseteq \widetilde{\mathfrak{S}}_n$ generated by $\{s_1, \dots, s_{n-1}\}$. The coset representatives of $\widetilde{\mathfrak{S}}_n / \mathfrak{S}_n$ are given by *affine grassmannian* elements $w \in \widetilde{\mathfrak{S}}_n$. These will be the minimal length representatives of each coset in $\widetilde{\mathfrak{S}}_n / \mathfrak{S}_n$. Equivalently, these are the w such that $w(1) < \cdots < w(n)$.

2.2 Types of Partitions

Definition 21: A partition λ is *k-bounded* if $\lambda_1 \leq k$.

The *hook length* of a cell in a tableau is the number of cells directly below and the number of cells directly to the right, including the cell itself.

A partition κ is an *r-core* if it has no cells of hook length equal to r. The *size* of κ, $|\kappa|_r$ is the number of cells of hook length less than r.

A k-skew diagram is a skew-partition κ/ρ such that no cell if κ/ρ has hook-length exceeding k and all the cells where ρ would be in the skew-partition have hook length exceeding k.

2.3 Bijections Fix some $n \geq 1$ and throughout let k such that $n = k+1$. All our examples will focus on $k = 3$, $n = 4$, and $m = 8$.

Proposition 22 ([Lam+14, Proposition 1.3]): There is a bijection between the set of $(k+1)$-cores κ with $|\kappa|_{k+1} = m$ and k-bounded partitions $\lambda \vdash m$.

The map sending a $(k+1)$-core to a k-bounded partition is done by simply removing the cells with hook length $\geq k+1$ and realigning the rows to form a partition. This is shown in Figure 1. The map sending a k-bounded partition to a $(k+1)$-core is computed by the following procedure. Starting from the smallest row in the diagram and moving upwards to the largest row, slide cells in the row (and thus the above rows) to the right until there are no hooks of length at least $k+1$ in the current row. This is shown in Figure 2.

Example 23:

Figure 1: Map sending the $(3+1)$-core $\kappa = (6,3,2,1)$ to the 3-bounded partition $\lambda = (3,2,2,1)$. Cells marked with a red X are the cells eliminated.

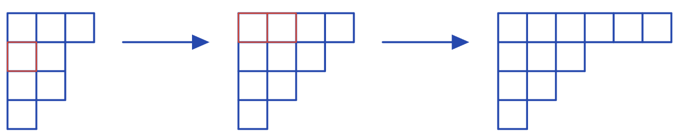

Figure 2: Map sending the 3-bounded partition $\lambda = (3,2,2,1)$ to the $(3+1)$-core $\kappa = (6,3,2,1)$. Cells outlined in red are the cells shifted to the right.

In fact, the skew-partition formed by taking the core κ and eliminating all cells with hook length $\geq k+1$ gives a k-skew diagram. Then "filling in" the k-skew diagram gives us a $(k+1)$-core. We record this bijection.

Proposition 24 ([LM04; LM05]): There is a bijection between the $(k+1)$-cores κ with $|\kappa|_{k+1} = m$ and the k-skew diagrams with m cells.

We also get a bijection from the k-skew diagrams to the k-bounded partitions. Left-alignment of the rows of the k-skew diagram gives us a k-bounded partition. To go from a k-bounded partition λ to a k-skew diagram, work from the smallest row to the largest row of of λ, adding rows to the left-most position that does not form any hooks of length exceeding $k+1$.

Proposition 25 ([LM04; LM05]): There is a bijection between the k-bounded partitions $\lambda \vdash m$ and the k-skew diagrams with m cells.

We see in Figure 3 that we have just built on the bijection stated in Proposition 22.

Example 26:

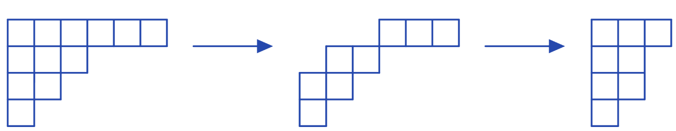

Figure 3: An example of the bijection between $(k+1)$-cores, k-skew diagrams, and k-bounded partitions.

Proposition 27 ([Lam+14, Proposition 1.9]): There is a bijection between the affine Grassmannian elements of $\widetilde{\mathfrak{S}}_n$ with length m and the $(k+1)$-cores κ with $|\kappa|_{k+1} = m$.

Definition 28: For a cell with coordinate (i, j) of a Young diagram, its *content* or *diagonal index* is the value $j - i$. For a fixed n, its *residue* modulo n is the content modulo n.

Example 29:

Figure 4: Content and residue modulo $3+1$ of the 3-bounded partition $\lambda = (3, 2, 2, 1)$.

We define an action of $\widetilde{\mathfrak{S}}_n$ on $(k+1)$-cores.

Definition 30: Let κ be a $(k+1)$-core and let $0 \le i \le n-1$. Then $s_i \cdot \kappa$ is the $(k+1)$-core such that

- If a cell with residue i can be added to κ such that the resulting diagram is still the diagram of a partition, add all such cells,
- If a cell with residue i can be removed from κ such that the resulting diagram is still the diagram of a partition, remove all such cells,
- Else, leave κ as-is.

Note that the first two conditions can never happen simultaneously.

We now give the bijection between $(k+1)$-cores κ of size m and affine grassmannian elements $w \in \widetilde{\mathfrak{S}}_n$ of length m. To go from an affine grassmannian element w of length m to a $(k+1)$-core κ of size m, use the above action of $\widetilde{\mathfrak{S}}_n$ starting from the empty $(k+1)$-core. This is shown in Figure 5. To go from a $(k+1)$-core κ to a word w, first convert κ to a k-bounded partition λ. Then read the residues of λ modulo n in each row right to left, from the bottom row to the top row, to get the sequence of generators to form w. This is shown in Figure 6.

Example 31: The affine grassmannian word $w = s_1 s_3 s_2 s_0 s_3 s_2 s_1 s_0 \in \widetilde{\mathfrak{S}}_4$ corresponds to the $(3+1)$-core $\kappa = (6, 3, 2, 1)$ by the following sequence of steps:

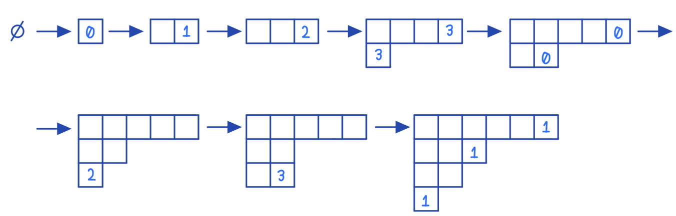

Figure 5: Action of $\widetilde{\mathfrak{S}}_4$ on $(3+1)$-cores to form the $(3+1)$-core $\kappa = (6, 3, 2, 1)$. The light-blue numbers indicate the residues of the cells being appended.

Below, we see that the residues of $\lambda = (3, 2, 2, 1)$ give the same affine grassmannian word w: Note that in the process of going from an affine grassmannian word w to a $(k+1)$-core κ and back to a word w', we will have that $w = w'$ in $\widetilde{\mathfrak{S}}_n$ but the actual words may be different. For example, since s_1 and s_3 commute, $w = s_1 s_3 s_2 s_0 s_3 s_2 s_1 s_0$ and $w' = s_3 s_1 s_2 s_0 s_3 s_2 s_1 s_0$ will produce the same $(3+1)$-core.

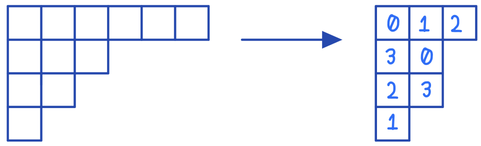

Figure 6: Prior example of converting κ to λ, with resides modulo 4 given in light-blue.

References

[BB05] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Vol. 231. Graduate Texts in Mathematics. New York: Springer-Verlag, 2005. 371 pp. ISBN: 978-3-540-44238-7. URL: http://link.springer.com/10.1007/3-540-27596-7.

[Ful96] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. 2nd ed. London Mathematical Society Student Texts 35. Cambridge: Cambridge University Press, 1996.

[GR26] Darji Grinberg and Victor Reiner. *Hopf Algebras in Combinatorics*. Online Notes. Version of May 14th 2026. 2026. URL: https://darijgrinberg.gitlab.io/algebra/HopfComb.pdf.

[Lam+14] Thomas Lam et al. *K-Schur Functions and Affine Schubert Calculus*. Vol. 33. Fields Institute Monographs. New York, NY: Springer, 2014. ISBN: 978-1-4939-0681-9 978-1-4939-0682-6. DOI: 10.1007/978-1-4939-0682-6. URL: https://link.springer.com/10.1007/978-1-4939-0682-6 (visited on 05/19/2026).

[LM04] Luc Lapointe and Jennifer Morse. *Tableaux on K+1-Cores, Reduced Words for Affine Permutations, and k-Schur Expansions*. arXiv.org. Feb. 19, 2004. DOI: 10.48550/arXiv.math/0402320.

[LM05] Luc Lapointe and Jennifer Morse. *A K-Tableau Characterization of k-Schur Functions*. arXiv.org. May 24, 2005. DOI: 10.48550/arXiv.math/0505519.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. 2nd ed. Oxford Mathematical Monographs. Oxford University Press, Mar. 1995. ISBN: 978-0-19-853489-1. DOI: 10.1093/oso/9780198534891.001.0001.

[Sta23] Richard Stanley. *Enumerative Combinatorics Volume 2*. 2nd ed. Vol. 2. 2 vols. Cambridge Studies in Advanced Mathematics 208. Cambridge: Cambridge University Press, 2023. ISBN: 978-1-009-26253-8. DOI: 10.1017/9781009262538.

[Stu08] Bernd Sturmfels. *Algorithms in Invariant Theory*. 2nd ed. Texts & Monographs in Symbolic Computation. Springer Vienna, 2008, p. vii 197. 197 pp. ISBN: 978-3-211-77417-5. DOI: 10.1007/978-3-211-77417-5. URL: http://link.springer.com/book/10.1007%2F978-3-211-77417-5.